ON THE GEOMETRY OF METRICS EMBEDDABLE IN THE REAL LINE

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ABSTRACT. Metrics that are embeddable in the real line have been widely studied, due to their many applications. Here, we study the subset of these metrics that arises when we impose a 'minimum separation' requirement, that the distance between any two points must be at least an (arbitrary) fixed positive number. These metrics have applications, for example, to graph layout problems such as the linear arrangement problem. We give several results on the geometry of this set.

We first show how the set of metrics is contained in its convex hull and characterize all unbounded one-dimensional extreme subsets of the convex hull combinatorially. Secondly, we give a combinatorial characterization of the set of unbounded edges of the closure of the convex hull. As a by-product, we obtain that the convex hull is closed if and only if $n \leq 3$.

Keywords: geometry of metrics, isometrically embeddable finite metric spaces, linear arrangements, spreading metrics

1. INTRODUCTION

We study three convex sets which are defined via certain (semi-)metrics on finite sets $[n] := \{1, \ldots, n\}$, for $n \geq 3$ an integer. Recall that a *semi-metric* on $[n]$ is a mapping $d \colon [n] \times [n] \to \mathbb{R}_+$ which satisfies the triangle inequality and $d(k, l) = d(l, k)$ for all $k, l \in [n]$. Note that $\mathbb{R}_+$ is the set of nonnegative reals. A *metric* is a semi-metric for which $d(k, l) > 0$ for all $k \neq l$.

In this paper we study metrics d on $[n]$ which are *embeddable in the real line.* This means that there exist real numbers $x_1, \ldots, x_n$ such that $d(k, l) = |x_k - x_l|$ for all $k, l \in [n]$. The set of semi-metrics that are embeddable in the real line, and their convex hull, have been widely studied, due to their many applications in diverse fields [5].

In this paper, we study the subset of these semi-metrics that arises when we impose an additional minimum "separation requirement", that $d(i, j) \geq \varepsilon$ for all $i \neq j$, where $\varepsilon > 0$ is a fixed quantity. We call such a metric an ε-separating metric. We will see below that ε can be set to 1 without loss of generality.

As well as studying these metrics themselves, we also study their convex hull, and the closure of their convex hull. As we explain below (Subsection 1.2), these

metrics and convex sets are of great interest, due to their applications to certain graph layout and facility layout problems.

This paper is strongly related to the earlier paper [1], which studies a family of metrics and convex sets that is closely related to ours emphasizing computational applicability: Whereas [1] is concerned mainly with deriving strong valid inequalities and using them as cutting planes, the present paper focuses attention on the extreme unbounded one-dimensional faces (i.e., the unbounded edges) of the convex sets.

The structure of the paper is as follows. In the remainder of the introduction, we formally define the metrics and sets of interest, and explain our motivation for studying them. Along the way, we review some of the relevant existing literature. Following that, in Section 3, we give basic properties of the set of semi-metrics embeddable in the real line. Here, a simple relationship between this set and the famous Permutahedron [9] is established. In Section 4, a closer investigation of this relationship leads to statements and proofs of structural results on the convex hull of ε-separating metrics. The section is closed by a combinatorial characterization of the unbounded extremal half lines of the convex hull of ε-separating metrics. Then, in Section 5, we prove the main contribution of this paper, the combinatorial characterization of the unbounded edges of the closure of the convex hull of ε-separating metrics. The characterizations are in terms of permutations and subsets of $[n]$. As a by-product, we obtain that Q_n is closed if and only if $n \leq 3$. In the final section, some conclusions are given.

1.1. **Basic terminology.** We recall some more facts, definitions and notation from [5]. The set of all semi-metrics is a polyhedral cone, called the *metric cone*. A semi-metric d is said to be a *cut* semi-metric if d has binary entries and there exists a set $S \subset [n]$ such that $d(i, j) = 1$ if and only if $|\{i, j\} \cap S| = 1$. The polyhedral cone defined by the conical combinations of the cut semi-metrics is called the *cut cone*, and denoted by C_n. The cut cone is strictly contained in the metric cone for $n \geq 5$. A semi-metric d is said to be ℓ_1-*embeddable* if there exist an integer m and points $x^1, \ldots, x^n \in \mathbb{R}^m$ such that $d(i, j) = \left| x^i - x^j \right|_1 := \sum_{i=1}^{m} |x_k^i - x_k^j|$. It is easy to show that the set of all ℓ_1-embeddable semi-metrics is equal to the cut cone. The ℓ_1-embeddable semi-metrics and the cut cone have been studied in great depth.

Now let E_n denote the set of all semi-metrics that are embeddable in the real line. Since these semi-metrics are a subset of the ℓ_1-embeddable metrics, E_n is a subset of the cut cone. It is easy to show that E_n is not convex. (In fact, we will see in Section 3 that E_n is the union of $n!/2$ simplicial cones, and that $E_n \setminus 0$ has the homotopy type of the real projective space of dimension $n - 2$.) It is also easy to show that E_n contains all of the cut metrics. (It suffices to set x_i to 0 if $i \in S$ and to 1 if $i \in [n] \setminus S$.) Therefore, the convex hull of E_n is again the cut cone.

We now let E_n^b denote the subset of E_n defined by the 1-separating metrics, i.e., those satisfying the "minimum separation" requirement $d(i, j) \geq 1$ for $i \neq j$. It is easy to see that E_n^b can be obtained from E_n by translating each of the $n!/2$ simplicial cones mentioned above. This makes the cones disjoint, since they no

longer have the origin in common. We will let Q_n denote the convex hull of E_n^b, and $\overline{Q_n}$ denote the closure of Q_n. (At this stage, it is not obvious whether or not Q_n is closed. We will see at the end of Section 5 that it is not closed in general.)
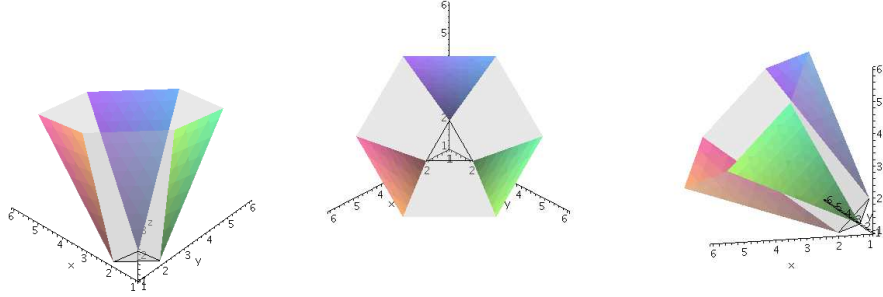

FIGURE 1. The convex set Q_3

Figure 1 shows three drawings of Q_3, in which the three coordinates x, y and z represent the values of $d(1, 2)$, $d(1, 3)$ and $d(2, 3)$, respectively. (Of course, the drawing is truncated, since Q_3 is unbounded.) Already, for $n = 3$, it is apparent that Q_n and $\overline{Q_n}$ are considerably more complicated than the cut cone: for $n = 3$, the cut cone is equal to the metric cone, and it therefore has only 3 facets and 3 edges.

1.2. **Motivation.** We regard E_n^b, Q_n and $\overline{Q_n}$ as natural and interesting sets to study in the own right. Nevertheless, an additional motivation for studying them comes from their application to an important class of combinatorial optimization problems, known as *graph layout* problems.

Graph layout problems call for a graph to be embedded in the real line (or occasionally in some other simple space, such as the plane), in order to minimize some specified objective function [6]. For example, in the *linear arrangement problem*, one is given a graph $G = (V, E)$, and one seeks a mapping $\pi\colon V \mapsto V$ such that $\sum_{\{i,j\}\in E} |\pi(i) - \pi(j)|$ is minimized. There is a natural class of metrics associated to such graph layout problems: the metrics d for which there exists a permutation $\pi\colon [n] \mapsto [n]$ such that $d(i, j) = |\pi(i) - \pi(j)|$. Let us denote by E_n^a this set of metrics.

By definition, the metrics in E_n^a are embeddable in the real line and satisfy the minimum separation requirement. Moreover, the metrics in E_n^a are precisely the metrics in E_n^b that satisfy the upper bounds $d(i, j) \leq n-1$ for all $\{i, j\} \subset [n]$. The convex hull of E_n^a (in fact of a more general class of metrics) was studied in [1], and several classes of valid and facet-inducing inequalities were derived. If we let P_n denote this convex hull, it is easy to show that:

- $\overline{Q_n}$ is the Minkowski sum of P_n and the cut cone.
- P_n is the unique bounded facet of $\overline{Q_n}$, defined by the equation

$$\sum_{\{i,j\}\subset[n]} d(i, j) = n(n + 1)(n - 1)/6.$$

- $\overline{Q_n}$ is a full-dimensional polyhedron, whereas P_n is not.
- Given any inequality that induces a facet of P_n, there is a unique inequality (up to scaling) that induces the same facet of P_n and induces a facet of $\overline{Q_n}$.

Thus, $\overline{Q_n}$ is a natural full-dimensional polyhedron containing P_n. Typically, full-dimensional polyhedra are easier to study than polyhedra that are not. In particular, viewing the polytope P_n as a face of $\overline{Q_n}$ allows to fix a natural "normal form" of facet-defining inequalities for P_n. Certain natural "lifting operations" can only be performed on inequalities if they are in this normal form. (These properties of $\overline{Q_n}$ were exploited in [1].)

Some authors have studied certain metrics that are related to graph layout problems. Even *et al.* [7] introduced the so-called *spreading metrics,* which are the metrics that satisfy the following *spreading inequalities*:

$$\sum_{j \in S \setminus \{i\}} d(i,j) \geq \frac{1}{4}|S|(|S|+2) \qquad (\forall i \in [n], \ \forall S \subset [n] \setminus \{i\}).$$

We remark that the metrics in E_n^b are spreading metrics. Indeed, one can show that the metrics in E_n^b satisfy the following slightly stronger inequalities:

$$\sum_{j \in S \setminus \{i\}} d(i,j) \geq \left\lfloor (|S|+1)^2/4 \right\rfloor \qquad (\forall i \in [n], \ \forall S \subset [n] \setminus \{i\}).$$

This in fact follows easily from observations of Chung [4], see also [1].

Some other metrics that have been introduced to form relaxations of graph layout problems include the *flow metrics* of Bornstein & Vempala [3] and the ℓ_2^2-*spreading metrics* of Charikar *et al.* [8]. It is not hard to show that the metrics in E_n^b are also flow metrics and ℓ_2^2-spreading metrics. We omit the details for the sake of brevity.

We have argued why the metrics we consider are important. As for (the closure of) their convex hull, we have sketched why studying it is related to certain combinatorial optimization problems. However, we would like to point the reader to another reason for investigating extreme rays of (the closure of) the convex hull of a set E, when the object of interest really is the set E itself. The reason is that the questions "how is a disjoint union of cones contained in its convex hull" and "how is its convex hull contained in the closure of the convex hull" shed light on the relative positions of the cones in space (we have mentioned above that the sets of metrics we consider are disjoint unions of simplicial cones). This can be visualized by considering the following three sets in 3-space. They all consist of two disjoint rays, but they differ by the presence or absence of extremal rays:

- $E := \big((1,0,0) + \mathbb{R}_+(1,0,0)\big) \ \uplus \ \big((0,1,0) + \mathbb{R}_+(0,1,0)\big)$;
- $E := \big((1,0,0) + \mathbb{R}_+(1,0,0)\big) \ \uplus \ \big((2,1,0) + \mathbb{R}_+(0,1,0)\big)$;
- $E := \big((1,0,0) + \mathbb{R}_+(1,0,0)\big) \ \uplus \ \big((0,1,1) + \mathbb{R}_+(0,1,0)\big)$.

2. PRELIMINARIES

For non-negative integers n, m, we denote by $\mathbb{M}(n \times m)$ the vector space of all real $n \times m$-matrices. Let $\mathbb{S}_n^0$ denote the vector space of real symmetric $n \times n$-matrices all of whose diagonal entries are equal to zero. We identify a semi-metric d on $[n]$ with a point D in $\mathbb{S}_n^0$ by letting $D_{k,l} = d(k, l)$ for all $k, l \in [n]$. The $\mathbb{S}_n^0$ is an $\binom{n}{2}$-dimensional subspace of $\mathbb{M}(n \times n)$, which is endowed with the natural inner product defined by

$$A \bullet B := \operatorname{tr}(A^\top B) = \sum_{k=1}^n \sum_{l=1}^n A_{k,l} B_{k,l}.$$

For a set $U \subset [n]$, we denote by $\complement U$ denotes the complement of the set U. Let $S(n)$ be the set of all permutations of $[n]$. We will identify a permutation $\pi \in S(n)$ with the point $(\pi(1), \ldots, \pi(n))^\top \in \mathbb{R}^n$. By $\imath_n := (1, \ldots, n)$ we denote the identity permutation in $S(n)$. We omit the index n when no confusion can arise. $\mathbf{1}$ is a column vector of appropriate length consisting of ones. Similarly $\mathbf{0}$ is a vector whose entries are all zero. If appropriate, we will use a subscript $\mathbf{1}_k$, $\mathbf{0}_k$ to identify the length of the vectors. The symbol $\mathbb{0}$ denotes an all-zeros matrix not necessarily square, and we also use it to say "this part of the matrix consists of zeros only." By $\mathbb{1}_n$ we denote the square matrix of order n whose (k, l)-entry is 1 if $k \neq l$ and 0 otherwise. As above we will omit the index n when appropriate.

Recall that a subset X of a convex set C is called *exposed,* if there exists a half space H containing C, such that the intersection of the bounding hyperplane of H with C is equal to X. In other words, X is exposed iff there exists a valid inequality for C such that X is the set of all points in C satisfying the inequality with equality. A subset X of a convex set C is called *extreme,* if $tc + (1-t)c' \in X$ for $c, c' \in C$ and $0 < t < 1$ implies $c, c' \in X$. Clearly, if X is exposed it is also extreme.

The permutahedron. The following well-known facts about the permutahedron can be found, for example, in [11].

Recall that the *permutahedron* is the convex hull of all permutations π when viewed as points in $\mathbb{R}^n$ as above. It is a zonotope, which means that it can be written as the Minkowski sum of line segments. We will use the notation

$$\Pi^{n-1} := \sum_{k=1}^{n-1} \sum_{l=k+1}^n \left[\tfrac{1}{2}(e_k - e_l), \tfrac{1}{2}(e_l - e_k) \right],$$

where e_i denotes the i-th unit vector in $\mathbb{R}^n$, and $[a, b]$ is the line segment joining two points. It is easy to see that, in $\mathbb{R}^n$, the "real" permutahedron is equal to a translation of Π^{n-1}:

$$\Pi^{n-1} + \tfrac{n+1}{2}\mathbf{1} = \operatorname{conv}\{\pi \mid \pi \in S(n)\}.$$

When written in this form, Π^{n-1} is full-dimensional in the linear subspace L^{n-1} of $\mathbb{R}^n$ defined by the equation $\sum_k x_k = 0$, it contains $0 \in L^{n-1}$ as a interior point (relative to L^{n-1}), and it is symmetric with respect to the origin: $\Pi^{n-1} =$

$-\Pi^{n-1}$. This makes Π^{n-1} easier to work with than the original definition of the permutahedron. We denote the vertex of the permutahedron Π^{n-1} corresponding to the permutation π by

$$v^\pi := \pi - \tfrac{n+1}{2}\mathbf{1}. \tag{1}$$

Note that we do not adhere to the convention which associates the permutation π^{-1} rather than π to the vertex $(\pi(1), \dots, \pi(n))^\top$ of the permutahedron, because it simplifies the notation for us. The facets of Π^{n-1} correspond to non-empty subsets $U \subsetneq [n]$. To be precise, a complete description of the permutahedron $\Pi^{n-1} + \frac{n+1}{2}\mathbf{1}$ is given by the inequalities

$$\sum_{j \in U} x_j \geq \binom{|U| + 1}{2}, \tag{2}$$

which are all facet-defining. From this, it is easy to see that Π^{n-1} is a simple polytope: a vertex of Π^{n-1} corresponding to a permutation π is contained in a facet corresponding to a set U if and only if

$$U = \{\pi^{-1}(1), \dots, \pi^{-1}(k)\}, \text{ where } k := |U|. \tag{3}$$

We say that a permutation π and a non-empty set $U \subsetneq [n]$ are *incident,* if (3) holds. Thus, incidence of permutations and subsets of $[n]$ reflects incidence of vertices and facets of the permutahedron and, of course, of facets and vertices of the polar of the permutahedron,

$$(\Pi^{n-1})^\triangle := \{a \in L^{n-1} \mid a^\top x \leq 1 \ \forall \, x \in \Pi^{n-1}\}.$$

The vertex of $(\Pi^{n-1})^\triangle$ corresponding to the facet of $\Pi^{n-1} + \frac{n+1}{2}\mathbf{1}$ defined by (2) is

$$a_U := \tfrac{2}{n(n-k)}\chi^{\complement U} - \tfrac{2}{kn}\chi^U. \tag{4}$$

Let π be a permutation and consider the facet of the polar $(\Pi^{n-1})^\triangle$ of the permutahedron corresponding to π. Since $(\Pi^{n-1})^\triangle$ is simplicial, if we start somewhere "on π" and "walk over" a particular ridge to a neighboring facet π', then a unique vertex "comes into sight." If U is the subset of $[n]$ corresponding to this vertex, we say that U is *over the ridge from π to π'* or just *over the ridge from π.* A set U is over the ridge from π if and only if it is of the form $U = \pi^{-1}([k-1] \cup \{k+1\})$, for a $k \in [n-1]$.

3. Basic properties of semi-metrics embeddable in the real line

In this section, we establish the basic background facts of this paper. In [5] a characterization of metrics which are ℓ_1-embedabble in dimension d via so-called d-nested families is given. Here, focusing on $d = 1$, we take a different approach which reveals the same structure in a more "continuous" way. Some preparation is necessary.

For a vector $x \in \mathbb{R}^n$, we let $M_{k,l}(x) := |x_k - x_l|$, and define a mapping

$$M : \mathbb{R}^n \to \mathbb{S}_n^0 : x \mapsto M(x) = \left(M_{k,l}(x)\right)_{\substack{k=1,\dots,n \\ l=1,\dots,n}}.$$

We can now write $C_n = \mathrm{cone}\{M(\chi^U) \mid U \subset n\}$, where χ^U is the characteristic vector of U in $\mathbb{R}^n$, i.e., the vector which has ones in the entries corresponding to elements of U and zeros otherwise. The matrix $M(\chi^U)$ corresponds to the cut semi-metric defined earlier. Moreover we have

$$E_n^b = \{M(x) \mid |x_k - x_l| \geq 1 \ \forall k \neq l\}. \tag{5}$$

Remark 3.1. Replacing the bound 1 by an arbitrary $\varepsilon > 0$ in (5) results in a dilation of the set E_n^b. Thus, this definition is sufficiently general for constant lower bounds.

The following lemma states some properties of M. For this, recall that the normal fan $\mathscr{N}$ of Π^{n-1} is a collection of cones N_F in L^{n-1}, where F ranges over the non-empty faces of Π^{n-1}. For any such F, the cone N_F is defined as the set of all vectors $c \in L^{n-1}$ for which the maximum of the linear function $x \mapsto c^\top x$ over Π^{n-1} is attained in all points of F. Clearly, $\mathscr{N}$ subdivides L^{n-1}. The normal fan of Π^{n-1} is equal to the face fan of the polar $(\Pi^{n-1})^\Delta$. We abbreviate $N_\pi := N_{\{v_\pi\}}$. This is an $(n-1)$-dimensional simplicial cone with apex 0 in L^{n-1}, which is generated by the extreme rays $\mathbb{R}_+ a_U$ where U ranges over all non-empty proper subsets of $[n]$ incident on π, and we have

$$N_\pi = \{x \in L^{n-1} \mid x_k \leq x_l \text{ for all } k, l \text{ with } \pi(k) < \pi(l)\}. \tag{6}$$

(We refer the reader to Chapter 7 in [11] for these facts.)

Remark 3.2. It is readily checked from the definition of v^π in (1) and the characterization of N_π in (6) that for each $\pi \in S(n)$ we have $v^\pi \in N_\pi$.

Lemma 3.3. *The mapping M has the following properties.*

(a) *We have $M(x + \xi \mathbf{1}) = M(x)$ for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$.*
(b) *For $x, y \in L^{n-1}$ we have*

$$M(x) = M(y) \quad \text{if, and only if,} \quad x = y \text{ or } x = -y.$$

(c) *The mapping M is linear on each of the cones N_π, and it is also injective there.*
(d) *For each π, the image of N_π under M is an $(n-1)$-dimensional simplicial cone with apex zero in $\mathbb{S}_n^0$, which is generated by the extreme rays $\mathbb{R}_+ M(\chi^U)$ where U ranges over all non-empty proper subsets of $[n]$ incident on π.*

Proof. *(a).* Obvious from the definition of M.

(b). This is an easy exercise which we leave to the reader.

(c). Linearity of M on N_π follows from the description of N_π in (6). The statement about the injectivity follows from (b).

(d). By the previous items, we know that $M(N_\pi)$ is the image of an $(n-1)$-dimensional simplicial cone with apex zero under an injective linear mapping. Moreover, as noted above, N_π is generated by the points a_U defined in (4), where U ranges over the $n-1$ sets incident to π. Since, by (a), $M(\chi^U) = M(a_U)$, the second part of the statement follows. □

Semi-metrics. We now obtain the following easy observations about the set E_n of all semi-metrics embeddable in $\mathbb{R}$.

Proposition 3.4. *We have* $E_n = M(L^{n-1})$. *Moreover, the following hold.*

   (a) E_n *is a simplicial fan consisting of* $n!/2$ *cones of dimension* $n - 1$.
   (b) $E_n \setminus \{0\}$ *is homeomorphic to the Cartesian product of* $\mathbb{R}$ *with the real projective space of dimension* $n - 2$.
   (c) $\operatorname{conv} E_n$ *is equal to the cut-cone.*

Proof. (a). The first statement is immediate from Lemma 3.3-(a). As for (a), we just note that M maps the normal fan of Π^{n-1} in L^{n-1} onto E_n identifying antipodal facets.

(b). The boundary of $(\Pi^{n-1})^\triangle$ is topologically an $(n-2)$-sphere, and M identifies antipodal points but is otherwise injective. Hence, $M((\Pi^{n-1})^\triangle)$ is homeomorphic to the real projective space of dimension $n - 2$. The statement now follows from the positive homogeneity of M, i.e., $M(\lambda x) = \lambda M(x)$ for all $\lambda > 0$, and by another reference to injectivity modulo antipodality.

(c). Is obvious.   □

We note that this proposition might as well have been proved directly from the characterization of metrics ℓ_1-embedabble in dimension d in [5], Prop. 4.2.2 and Lemma 11.1.3. Item (b), for example, would require to identify that $E_n \setminus \{0\} \cong \mathbb{R} \times K$, where K is a simplicial complex which can be readily recognized to be a barycentric subdivision of an $(n - 2)$-simplex after identification of antipodal points.

4. THE CONVEX HULL Q_n OF E_n^b

Recall that we have defined

$$Q_n := \operatorname{conv} E_n^b \quad \text{and} \quad P_n := \operatorname{conv}\big\{M(\pi) \mid \pi \in S(n)\big\}.$$

Remark 4.1. For a permutation π, recall the definition of its permutation matrix E_π which is an $n \times n$-matrix which has, for every l, a unique non-zero entry in the lth column, namely a one in the $\pi(l)$th row. It is clear that $M \mapsto E_\pi^\top M E_\pi$ is a linear isomorphism on $\mathbb{S}_n^0$ which maps P_n onto P_n and Q_n onto Q_n. If $\sigma \in S(n)$ and $x \in \mathbb{R}^n$, letting $(x \circ \sigma)_j := x_{\sigma(j)}$ for all $j \in [n]$, we have $E_\sigma^\top M(x) E_\sigma = M(x \circ \sigma)$.

An immediately consequence of this remark is that the vertices of P_n are exactly the matrices $M(\pi)$ for π a permutation in $S(n)$ (which is proven in [1]).

In view of Lemma 3.3-(b), we define the *antipodal* permutation of $\pi \in S(n)$ by

$$\pi^- := (n+1) \cdot \mathbf{1} - \pi.$$

Note that $v^{\pi^-} = -v^\pi$. From (a) and (b) we know that $M(\pi) = M(v^\pi) = M(-v^\pi) = M(\pi^-)$. Note also that $M(\chi^U) = M(\chi^{\complement U})$. One might want to call $\complement U$ the antipode of U because $a_U = -a_{\complement U}$.

When, for ease of notation, we let

$$R_n := \big\{x \in L^{n-1} \mid |x_k - x_l| \geq 1 \,\forall k \neq l\big\},$$

then, by Lemma 3.3-(a), we have $E_n^b = M(R_n)$ and $Q_n = \text{conv}\, M(R_n)$. Since L^{n-1} is the union of the cones N_π when π ranges over all permutations, we know that

$$E_n^b = M(R_n) = \bigcup_\pi M(R_n \cap N_\pi). \tag{7}$$

In the following lemma, we show that the sets $M(R_n \cap N_\pi)$ can be replaced by the translated cones $M(\pi) + M(N_\pi)$.

Lemma 4.2. *For every permutation π of $[n]$ we have*

$$R_n \cap N_\pi = v^\pi + N_\pi.$$

Proof. We first show $R_n \cap N_\pi \subset v^\pi + N_\pi$. For this, let x be any element in N_π with $|x_k - x_l| \geq 1$. We show that $y := x - v^\pi \in N_\pi$. To do this, we check whether the inequalities in (6) are all satisfied. For any j, j' with $\pi(j) < \pi(j')$, since $x \in N_\pi$, we know that $x_j \leq x_{j'}$, and because $x \in R_n$, we can strengthen this to $x_{j'} - x_j \geq 1$. For any k, l with $\pi(l) - \pi(k) =: r > 0$, if $j_0, \ldots, j_r$ are in $[n]$ with $\pi(k) = \pi(j_0) < \cdots < \pi(j_r) = \pi(l)$, we can telescope

$$x_l - x_k = \sum_{i=0}^{r-1} (x_{j_{i+1}} - x_{j_i}) \geq \sum_{i=0}^{r-1} 1 = \pi(l) - \pi(k),$$

and conclude that

$$y_l - y_k = x_l - x_k - (v_l^\pi - v_k^\pi) = x_l - x_k - (\pi(l) - \pi(k)) \geq 0.$$

Secondly, we show that $v^\pi + N_\pi \subset R_n \cap N_\pi$. Let $x \in N_\pi$. Now $x, v^\pi \in N_\pi$ (cf. Remark 3.2) implies $x + v^\pi \in N_\pi$ because N_π is a convex cone. For any k, l with $\pi(k) < \pi(l)$, since $x_k \leq x_l$, we compute $x_l + v_l^\pi - (x_k + v_k^\pi) \geq v_l^\pi - v_k^\pi = \pi(k) - \pi(l) \geq 1$. This implies $|x_i + v_i^\pi - (x_j + v_j^\pi)| \geq 1$ for all $i \neq j$, and hence $x + v^\pi \in R_n$. This proves $v^\pi + x \in R_n \cap N_\pi$. $\square$

Now we come to the structural results for E_n^b. The following proposition and its corollary are the basis of our work with E_n^b, Q_n and $\overline{Q_n}$. Together with Proposition 4.6 in the next subsection, they answer the question of how E_n^b is contained in its convex hull.

Proposition 4.3. *The set E_n^b is the union of $n!/2$ pairwise disjoint $(n-1)$-dimensional simplicial cones of the form $M(\pi) + M(N_\pi)$, where N_π is the normal cone of Π^{n-1} in L^{n-1} at the vertex v^π. Two cones $M(\pi) + M(N_\pi)$ and $M(\pi') + M(N_{\pi'})$ are identical if π' and π are identical or antipodal; otherwise they are disjoint.*

Proof. From equation (7), using $M(\pi) = M(v^\pi)$, the fact that M is linear on N_π by Lemma 3.3-(c), and the previous Lemma 4.2, we obtain

$$M(R_n \cap N_\pi) = M(v^\pi + N_\pi) = M(v^\pi) + M(N_\pi) = M(\pi) + M(N_\pi).$$

This implies $E_n^b = \bigcup_\pi (M(\pi) + M(N_\pi))$. Clearly, the set $M(N_\pi)$, is a simplicial cone because N_π is a simplicial cone and M is linear and injective on N_π.

Since $M(v^\pi + N_\pi) = M(-(v^\pi + N_\pi)) = M(v^{\pi^-} + N_{\pi^-})$, the number of distinct cones is at most $n!/2$. Using the definition of R_n and the outer descriptions

of the cones N_π in (6), we see that the $n!$ sets $R_n \cap N_\pi$ are all disjoint and the intersection of $R_n \cap N_\pi$ with $-(R_n \cap N_{\pi'})$ is non-empty if and only if $\pi' = \pi^-$. By Lemma 3.3-(b), this implies that two cones $M(\pi) + M(N_\pi)$ and $M(\sigma) + M(N_\sigma)$ are identical if π and σ are equal or antipodal, and that they are disjoint in any other case. Thus, there are $n!/2$ pairwise disjoint cones. $\square$

We note some consequences of the proposition.

Corollary 4.4.

  (a) *Q_n is the convex hull of all the half-lines $M(\pi) + \mathbb{R}_+ M(\chi^U)$, where π is a permutation of $[n]$, and U is a non-empty proper subset of $[n]$, such that π and U are incident.*
  (b) *The closure $\overline{Q_n}$ of Q_n is equal to the Minkowski sum $P_n + C_n$.*
  (c) *Q_n is a full-dimensional unbounded convex set.*
  (d) *Q_n contains P_n as an exposed subset: the inequality $\mathbb{1}_n \bullet X \geq 2\binom{n+1}{3}$ is valid for Q_n and satisfied with equality by the points $M(\pi)$, $\pi \in S(n)$.*
  (e) *P_n is the only bounded facet of the closure $\overline{Q_n}$ of Q_n.*

Proof. The proofs are easy consequences of the proposition. We sketch the arguments.

(a). Follows from Proposition 4.3 because, by Lemma 3.3-(d) the extreme rays of $M(\pi) + M(N_\pi)$ are just the half-lines $M(\pi) + \mathbb{R}_+ M(\chi^U)$ for U incident to π.

(b). It is obvious from (a) that $Q_n \subseteq P_n + C_n$. The following elementary argument shows that $P_n + C_n \subset \overline{Q_n}$. Let y be a vertex of P_n and R an extreme ray of C_n. We have to make sure that $x + R \subset \overline{Q_n}$. Clearly, $x \in Q_n$, and there exists a $y \in Q_n$ such that $y + R \subset Q_n$. Thus, the infinite open rectangle $\text{conv}\{x,y\} + R$ is contained in Q_n. Consequently, the closure of the rectangle is contained in $\overline{Q_n}$, but $x + R$ is contained in the closure of $\text{conv}\{x,y\} + R$.

(c). From (b) because C_n is full-dimensional.

(d). Directly from (a).

(e). Amaral & Letchford [1] proved that the polytope P_n has dimension $\binom{n}{2} - 1$ and that the equation

$$\mathbb{1}_n \bullet X = 2\binom{n+1}{3}$$

holds for all $X \in P_n$. Since $\overline{Q_n} = P_n + C_n$ is full-dimensional and C_n is contained in the half-space defined by $\mathbb{1} \bullet X \geq 0$, it follows that P_n is a facet of $\overline{Q_n}$. Any other facet of $\overline{Q_n}$ can contain only a proper subset of the vertices of P_n. Hence it must be unbounded. $\square$

Unbounded extremal subsets of Q_n. We now investigate *how* the simplicial cones $M(\pi) + M(N_\pi)$ are subsets of Q_n. In Fig. 1, it can be seen that in the case $n = 3$, the three cones are faces of Q_3 (recall that Q_3 is a polyhedron, which means that we can safely speak of faces). In the following proposition, we show that this is the case for all n, and we also characterize the extremal half-lines of Q_n. This will be useful in comparing Q_n with its closure: We will characterize the

unbounded edges issuing from each vertex for the polyhedron $\overline{Q_n} = P_n + C_n$ in the next section.

We are dealing with an unbounded convex set of which we do not know whether it is closed or not. (In fact, we will show in the next section that Q_n is almost never closed). For this purpose, we supply the following fact for easy reference.

Fact 4.5. For $k = 1, \ldots, m$ let K_k be a (closed) polyhedral cone with apex x_k. Suppose that the K_k are pairwise disjoint and define $S := \biguplus_{k=1}^{m} K_k$. Let x, y be vectors such that $x + \mathbb{R}_+ y$ is an extremal subset of $\operatorname{conv}(S)$. It then follows that there exists a $\lambda_0 \in \mathbb{R}_+$ and a k such that $x + \lambda y \in K_k$ for all $\lambda \geq \lambda_0$. Since $x + \mathbb{R}_+ y$ is extremal, this implies that there exists a $\lambda_1 \in \mathbb{R}_+$ such that $x_k = x + \lambda_1 y$ and $x_k + \mathbb{R}_+ y = \{x + \lambda y \mid \lambda \geq \lambda_1\}$ is an extreme ray of the polyhedral cone K_k.

Proposition 4.6.

    (a) *The extreme points of Q_n are precisely the vertices of P_n, which are of the form $M(\pi)$, for $\pi \in S(n)$. They are also exposed.*

    (b) *For every π, each face of the cone $M(\pi) + M(N_\pi)$ is an exposed subset of Q_n.*

    (c) *The unbounded one dimensional extremal sets of Q_n are exactly the defining half-lines. In other words, every half-line $X + \mathbb{R}_+ Y$ which is an extremal subset of Q_n is of the form $M(\pi) + \mathbb{R}_+ M(\chi^U)$ for a $\pi \in S(n)$ and a set U incident to π. In particular, for every vertex $M(\pi)$ of Q_n, the unbounded one-dimensional extremal subsets of Q_n containing $M(\pi)$ are in bijection with the non-empty proper subsets of $[n]$ incident to π. Thus there are precisely $n - 1$ of them.*

Proof. *(a).* This statement follows from Corollary 4.4, items (a) and (d).

(b). By the remark about the symmetry of Q_n at the beginning of this section, it is sufficient to treat the case $\pi = \imath := (1, \ldots, n)^\top$, the identity permutation.

Consider the matrix

$$
C := \begin{pmatrix}
0 & 1 & & & & -1 \\
1 & 0 & 1 & & \mathbb{0} & \\
 & 1 & & & & \\
 & & & \ddots & & \\
 & \mathbb{0} & & & 1 & \\
 & & & 1 & 0 & 1 \\
-1 & & & & 1 & 0
\end{pmatrix} \in \mathbb{S}_n^0.
$$

It is easy to see that the minimum over all $C \bullet M(\pi)$, $\pi \in S(n)$, is attained only in $\pi = \imath, \imath^-$ with the value 0. Moreover, for any non-empty proper subset U of $[n]$, we have $C \bullet M(\chi^U) = 0$ if U is incident to $\imath$ and $C \bullet M(\chi^U) > 0$ otherwise. Hence, we have that $M(\imath) + M(N_\imath)$ is equal to the set of all points in Q_n which satisfy the valid inequality $C \bullet X \geq 0$ with equality. Out of this matrix C we will now construct a matrix C' and a right hand side such that only some of the subsets incident to $\imath$ fulfill the inequality with equality. To do so let $U_1, \ldots, U_r$ be any set of subsets of $[n]$ incident to $\imath$. Increasing for each $i = 1, \ldots, r$ the matrix entries $C_{\max U_i, \max U_i + 1}$ and $C_{\max U_i + 1, \max U_i}$ by one gives an inequality $C' \bullet X \geq 0$ which is valid for Q_n and such that the set of all points of Q_n which are satisfied with equality is precisely the face of $M(\imath) + M(N_\imath)$ generated by the

half-lines $M(\imath) + \mathbb{R}_+ M(\chi^U)$, for which U is incident to π and satisfies $U \neq U_i$ for all $i = 1, \ldots, r$.

(c). That the defining half lines are extremal has just been proved in (b). The converse statement follows from (a) and Fact 4.5. $\qquad\qquad\square$

Remark 4.7. We note that in the proof of part (a) of the proposition, what we have actually proven is that for every set $\{W_1, \ldots, W_r\}$ of non-empty proper subsets of $[n]$ incident on π, there is a matrix C such that the minimum $C \bullet M(\sigma)$ over all $\sigma \in S(n)$ is attained solely in π and π^-, and that $C \bullet M(\chi^{W'}) \geq 0$ for every non-empty proper subset U' of $[n]$ where equality holds precisely for the sets W_i and their complements. This implies that $M(\pi) + \mathrm{cone}\{M(\chi^{W_1}), \ldots, M(\chi^{W_r})\}$ is a face of the polyhedron $\overline{Q_n} = P_n + C_n$.

5. Unbounded edges in the closure $\overline{Q_n}$ of the convex hull

We have just identified some unbounded edges of $\overline{Q_n} = P_n + C_n$ starting at a particular vertex $M(\pi)$ of this polyhedron. We now set off to characterize *all* unbounded edges of $\overline{Q_n}$. Clearly, the unbounded edges are of the form $M(\pi) + \mathbb{R}_+ M(\chi^U)$, but not all these half-lines are edges. For a permutation π and a non-empty subset $U \subsetneq [n]$, we say that $M(\pi) + \mathbb{R}_+ M(\chi^U)$ is the half-line *defined by the pair $\pi \nearrow U$*. In this section, we characterize the pairs $\pi \nearrow U$ which have the property that the half-lines they define are edges.

Theorem 5.1. *For all $n \geq 3$, the unbounded edges of $\overline{Q_n}$ are precisely the half-lines defined by those pairs $\pi \nearrow U$, for which neither U nor $\complement U$ is over the ridge from π.*

Major parts of the proof of this theorem work in an inductive fashion by reducing to the case when $n \in \{3, 4, 5, 6\}$. We will present the cases $n = 3$ and $n = 4$ as examples, which also helps motivating the definitions we require for the proof.

We will switch to a more "visual" notation of the subsets of $[n]$ by identifying a set U with a "word" of length n over $\{0, 1\}$ having a 1 in the jth position iff $j \in U$ — it is just the row-vector $(\chi^U)^\top$.

Example 5.2 (Unbounded edges of $\overline{Q_3}$). We deal with the case $n = 3$ "visually" by regarding Fig. 1. There are two edges starting at each vertex. In fact, with some computation, it can be seen that the unbounded edges containing $M(\imath)$ are

$$M\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) + \mathbb{R}_+ M\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1&2\\1&0&1\\2&1&0\end{smallmatrix}\right) + \mathbb{R}_+\left(\begin{smallmatrix}0&1&1\\1&0&0\\1&0&0\end{smallmatrix}\right), \quad \text{and}$$

$$M\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) + \mathbb{R}_+ M\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1&2\\1&0&1\\2&1&0\end{smallmatrix}\right) + \mathbb{R}_+\left(\begin{smallmatrix}0&0&1\\0&0&1\\1&1&0\end{smallmatrix}\right); \quad \text{while}$$

$$M\left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right) + \mathbb{R}_+ M\left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1&2\\1&0&1\\2&1&0\end{smallmatrix}\right) + \mathbb{R}_+\left(\begin{smallmatrix}0&1&0\\1&0&1\\0&1&0\end{smallmatrix}\right)$$

is not an edge. This agrees with Proposition 4.6, because the sets 100 and 110 are incident to $\imath$, while 101 and 010 are not. Moreover, the set 101 is over the ridge from $\imath$ and 010 is its complement. Thus, Theorem 5.1 is true for the special case when $\pi = \imath$. For the other permutations, the easiest thing to do is to use symmetry. We describe this in the next remark.

Remark 5.3. For every $\sigma, \pi \in S(n)$ and $U \subset [n]$ we have the following.

(a) By Remark 4.1, the pair $\pi \diagup U$ defines an edge of $\overline{Q_n}$ if and only if the pair $\pi \circ \sigma \diagup \sigma^{-1}(U)$ defines an edge of $\overline{Q_n}$.
(b) U is incident to π if and only if $\sigma^{-1}(U)$ is incident to $\pi \circ \sigma$.
(c) U is over the ridge from a permutation π if and only if $\sigma^{-1}(U)$ is over the ridge from $\pi \circ \sigma$.
(d) $\complement U$ is over the ridge from a permutation π if and only if U is over the ridge from π^-.

Proof. The last three statements are most easily realized by noting that $x \mapsto x \circ \sigma$ is a linear isomorphism of L^{n-1} taking $(\Pi^{n-1})^\triangle$ onto itself in such a way that the facet corresponding to a permutation π is mapped to the facet corresponding to $\pi \circ \sigma$, and the vertex corresponding to a set U is mapped to the vertex corresponding to the set $\sigma^{-1}(U)$. $\square$

We now give the first general result as a step towards the proof of Theorem 5.1.

Lemma 5.4. *If $\pi \in S(n)$ and $U \subset [n]$ is over the ridge from π, then the half-line $M(\pi) + \mathbb{R}_+ M(\chi^U)$ defined by the pair $\pi \diagup U$ is not an edge of $\overline{Q_n}$.*

Proof. By the above remarks on symmetry, it is sufficient to prove the claim for the identical permutation $\imath \in S(n)$. Consider a $k \in [n-1]$, and let $\pi' := \langle k, k+1 \rangle$ be the transposition exchanging k and $k+1$, and let $U := [k-1] \cup \{k+1\}$. Then a little computation shows that $M(\chi^U)$ can be written as a conic combination of vectors defining rays issuing from $M(\imath)$ as follows:

$$M(\chi^U) = M(\chi^{[k]}) + \big(M(\pi') - M(\imath)\big).$$

Hence $M(\imath) + \mathbb{R}_+ M(\chi^U)$ is not an edge. $\square$

Note that by applying Remark 5.3, the Lemma 5.4 implies that if $\complement U$ is over the ridge from π, then the pair $\pi \diagup \complement U$ does not define an edge of $\overline{Q_n}$.

Before we proceed, we note the following easy consequence of Farkas' Lemma.

Lemma 5.5. *The following are equivalent:*

(i) *The half-line $M(\imath) + \mathbb{R}_+ M(\chi^U)$ defined by the pair $\imath \diagup U$ is an edge of $\overline{Q_n}$.*
(ii) *There exists a matrix D satisfying the following constraints:*

$$D \bullet M(\pi) > D \bullet M(\imath) \qquad\qquad \forall\, \pi \neq \imath, \imath^-, \qquad\qquad (8a)$$

$$D \bullet M(\chi^{U'}) > D \bullet M(\chi^U) = 0 \qquad\qquad \forall\, U' \neq U, \complement U. \qquad\qquad (8b)$$

(iii) *There exists a matrix C satisfying*

$$C \bullet M(\pi) \;\; \geq C \bullet M(\imath) \qquad \forall\, \pi \neq \imath, \imath^-, \qquad\qquad (9a)$$

$$C \bullet M(\chi^{U'}) \geq 0 \qquad\qquad \forall\, U' \neq U, \complement U, \qquad\qquad (9b)$$

$$C \bullet M(\chi^U) \;\; < 0. \qquad\qquad\qquad\qquad\qquad\qquad (9c)$$

$\square$

Condition (9) is easier to check for individual matrices, but condition (8) will be needed in a proof below.

We move on to the next example which both provides some cases needed for the proof of Theorem 5.1 and motivates the following definitions.

Let U be a subset of $[n]$ and consider its representation as a word of length n. We say that a maximal sequence of consecutive 0s in this word is a *valley* of U. In other words, a valley is an inclusion wise maximal subset $[l, l + j] \subset \complement U$. Accordingly, a maximal sequence of consecutive 1s is called a *hill*. A valley and a hill meet at a *slope*. Thus the number of slopes is the number of occurrences of the patterns 01 and 10 in the word, or in other words, the number of $k \in [n-1]$ with $k \in U$ and $k + 1 \notin U$ or vice versa. If all valleys and hills of a subset U of $[n]$ consist of only one element (as for example in 10101) or, equivalently, if U has the maximal possible number $n - 1$ of slopes, or, equivalently, if U consists of all odd or all even numbers in $[n]$, we speak of an *alternating* set.

Example 5.6 (Unbounded edges of $\overline{Q_4}$). We consider the edges of $\overline{Q_4}$ containing $M(\imath) = M(\imath^-)$ (this is justified by Remark 5.3). We distinguish the sets U by their number of slopes. Clearly, a set U with a single slope is incident either to $\imath$ or to $\imath^-$, and we have already dealt with that case in Remark 4.7 The following sets have two slopes: 0100, 0110, 0010, 1011, 1001, and 1101. We only have to consider 1011, 1001, and 1101, because the others are their complements. The first one, 1011, is over the ridge from $\imath^-$, and the last one, 1101, is over the ridge from $\imath$, so we know that the pairs $\imath \nearrow 1011$ and $\imath \nearrow 1101$ do not define edges of $\overline{Q_4}$ by Lemma 5.4. For the remaining set with two slopes, 1001, the following matrix satisfies property (9) with C replaced by C^{1001} and U by 1001:

$$
C^{1001} := \begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 3 & -2 \\ -2 & 3 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}.
$$

The two alternating sets (i.e., sets with tree slopes) are 1010 and 0101, which are over the ridge from $\imath$ and $\imath^-$ respectively. This concludes the discussion of $\overline{Q_4}$.

Having settled some of the cases for small values of n, we give the result by which the reduction to smaller n is performed, which is an important ingredient for settling Theorem 5.1. The following lemma shows that unbounded edges of $\overline{Q_n}$ can be "lifted" to a larger polyhedron $\overline{Q_{n+k}}$.

Lemma 5.7. *Let U_0 be a non-empty proper subset of $[n]$ whose word has the form $a1b$ for two (possibly empty) words a, b. For any $k \geq 0$ define the subset U_k of $[n+k]$ by its word*

$$
U_k := a \underbrace{1 \ldots 1}_{k+1} b.
$$

If the pair $\imath_n \nearrow U_0$ defines an edge of $\overline{Q_n}$, then the pair $\imath_{n+k} \nearrow U_k$ defines an edge of $\overline{Q_{n+k}}$.

Note that the lemma also applies to consecutive zeroes, by exchanging the respective set by its complement.

Proof. Let $C \in \mathbb{S}_n^0$ be a matrix satisfying conditions (9) for $U := U_0$. Fix $k \geq 1$ and let $n' := n + k$. We will construct a matrix $C' \in \mathbb{S}_{n'}^0$ satisfying (9) for $U := U_k$. For a "big" real number $\omega \geq 1$ define a matrix $B_\omega \in \mathbb{S}_{k+1}^0$ whose entries are zero except for those connecting j and $j + 1$, for $j \in [k]$:

$$
B_\omega := \begin{pmatrix} 0 & \omega & & & & \\ \omega & 0 & \omega & & \mathbb{0} & \\ & \omega & & & & \\ & & & \ddots & & \\ & & & & & \omega \\ & \mathbb{0} & & \omega & 0 & \omega \\ & & & & \omega & 0 \end{pmatrix}.
$$

We use this matrix to put a heavy weight on the "path" which we "contract." For our second ingredient, let l_a denote the length of the word a and l_b the length of the word b (note that $l_a = 0$ and $l_b = 0$ are possible). Then we define

$$
B_- := \begin{pmatrix} +1 & \ldots & +1 \\ \mathbf{0}_{k-1} & \ldots & \mathbf{0}_{k-1} \\ -1 & \ldots & -1 \end{pmatrix} \in \mathbb{M}((k+1) \times l_a) \qquad \text{and}
$$

$$
B_+ := \begin{pmatrix} -1 & \ldots & -1 \\ \mathbf{0}_{k-1} & \ldots & \mathbf{0}_{k-1} \\ +1 & \ldots & +1 \end{pmatrix} \in \mathbb{M}((k+1) \times l_b),
$$

where $\mathbf{0}_{k-1}$ stands for a column of $k - 1$ zeros. Putting these matrices together we obtain an $n' \times n'$-matrix B:

$$
B := \begin{pmatrix} \mathbb{0} & B_-^\top & \mathbb{0} \\ B_- & B_\omega & B_+ \\ \mathbb{0} & B_+^\top & \mathbb{0} \end{pmatrix}.
$$

Now it is easy to check that for any $\pi' \in S(n')$ we have $B \bullet M(\pi') \geq B \bullet M(\imath)$. Moreover let $\pi' \in S(n')$ satisfy $B \bullet M(\pi') < B \bullet M(\imath) + 1$. By exchanging π' with π'^-, we can assume that $\pi'(1) < \pi'(n')$. It is easy to see that such a π' then has the following "coarse structure"

$$
\begin{aligned}
\pi'([l_a]) &\subset [l_a] \\
\pi'([n'] \setminus [n' - l_b]) &\subset [n'] \setminus [n' - l_b] \\
\pi'(j) &= j \quad \forall\, j \in \{l_a + 1, \ldots, l_a + k + 1\}.
\end{aligned} \tag{10}
$$

Thus the matrix B enforces that the "coarse structure" of a $\pi' \in S(n')$ minimizing $B \bullet M(\pi')$ coincides with $\imath$. We now modify the matrix C to take care of the "fine structure". For this, we split C into matrices $C_{11} \in \mathbb{S}_{l_a}^0$, $C_{22} \in \mathbb{S}_{l_b}^0$, $C_{12} \in \mathbb{M}(l_a \times l_b)$, $C_{21} = C_{12}^\top \in \mathbb{M}(l_b \times l_a)$, and vectors $c \in \mathbb{R}^{l_a}$, $d \in \mathbb{R}^{l_b}$ as follows:

$$
C = \begin{pmatrix} C_{11} & c & C_{12} \\ c^\top & 0 & d^\top \\ C_{21} & d & C_{22} \end{pmatrix}.
$$

Then we define the "stretched" matrix $\check{C} \in \mathbb{S}_{n'}^0$ by

$$\check{C} := \begin{pmatrix} C_{11} & c & \mathbb{0} & \mathbf{0} & C_{12} \\ c^\top & 0 & & 0 & \mathbf{0}^\top \\ & & & & \\ \mathbb{0} & & \mathbb{0} & & \mathbb{0} \\ & & & & \\ \mathbf{0}^\top & 0 & & 0 & d^\top \\ C_{21} & \mathbf{0} & \mathbb{0} & d & C_{22} \end{pmatrix},$$

where the middle $\mathbb{0}$ has dimensions $(k-1) \times (k-1)$. Finally we let $C' := B + \varepsilon\check{C}$, where $\varepsilon > 0$ is small. We show that C' satisfies (9).

We first consider $C' \bullet M(\chi^{U'})$ for non-empty subsets $U' \subsetneq [n']$. Note that, if U' contains $\{l_a + 1, \ldots, l_a + k + 1\}$, then for $U := U \setminus \{l_a + 1, \ldots, l_a + k + 1\}$, we have $C' \bullet M(\chi^{U'}) = C \bullet M(\chi^U)$. Thus we have $C' \bullet M(\chi^{U_k}) = C \bullet M(\chi^{U_0}) < 0$ proving (9c) for C' and U_k. For every other U' with $C' \bullet M(\chi^{U'}) < 0$, if ω is big enough, then either U' or $\complement U'$ contains $\{l_a + 1, \ldots, l_a + k + 1\}$, and w.l.o.g. we assume that U' does. By (9b) applied to C and U, we know that this implies $U = U_0$ or $U = \complement U_0$ and hence $U' = U_k$ or $\complement U' = U_k$. Thus, (9b) holds for C' and U_k.

Second, we address the permutations. To show (9a), let $\pi' \in S(n)$ be given which minimizes $C' \bullet M(\pi')$. Again, by replacing π' by π'^- if necessary, we assume $\pi'(1) < \pi'(n')$ w.l.o.g. If ε is small enough, we know that π' has the coarse structure displayed in (10). This implies that we can define a permutation $\pi \in S(n)$ by letting

$$\pi(j) := \begin{cases} \pi'(j) & \text{if } j \in [l_a], \\ \pi'(j) = j & \text{if } j = l_a + 1, \\ \pi'(j - k) + k & \text{if } j \in [n] \setminus [l_a + 1]. \end{cases}$$

An easy but lengthy computation (see [10] for the details) shows that

$$\begin{aligned} C' \bullet M(\pi') - C' \bullet M(\imath_{n'}) &= C \bullet M(\pi) + (k-1) \cdot C \bullet \begin{pmatrix} \mathbb{0}_{l_a \times l_a} & \mathbb{1} \\ \mathbb{1} & \mathbb{0}_{l_b \times l_b} \end{pmatrix} \\ &\quad - \left(C \bullet M(\imath_n) + (k-1) \cdot C \bullet \begin{pmatrix} \mathbb{0}_{l_a \times l_a} & \mathbb{1} \\ \mathbb{1} & \mathbb{0}_{l_b \times l_b} \end{pmatrix} \right) \\ &= C \bullet M(\pi) - C \bullet M(\imath_n) \geq 0. \end{aligned}$$

Thus (9a) holds. $\qquad\square$

Example 5.8. We give an example for the application of Lemma 5.7. For $n = 5$, consider the half-line defined by the pair $\imath \nearrow 11001$. The set 11001 can be reduced to 1001 by contracting the hill $1-2$. To do so we set

$$C^{11001} := \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & -2 & 3 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \omega & -1 & -1 & -1 \\ \omega & 0 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

for a small $\varepsilon > 0$ and a big $\omega \geq 1$.

Completion of the proof of Theorem 5.1. After these preparations we can tackle the proof of the theorem.

Proof of Theorem 5.1. By Remark 5.3, we only need to consider $\pi = \imath$. We distinguish the sets U by their numbers of slopes.

One slope. This is equivalent to U or $\complement U$ being incident to $\imath$. We have treated this case in Remark 4.7 of the previous section.

Two slopes. The complete list of all possibilities, up to complements, and how they are dealt with is summarized in Table 5. In this table, 0 stands for a valley consisting of a single zero while $0 \ldots 0$ stands for a valley consisting of at least two zeros (the same with hills). The matrices for the reduced words satisfying (9) can be found in the appendix on page 20. The condition (9) can be verified by some case distinctions.

	Word		Edge?	Why?
Hill 1	Valley	Hill 2		
1	0	1	no	over the ridge from $\imath$
1	0	$1 \ldots 1$	no	over the ridge from $\imath^-$
1	$0 \ldots 0$	1	yes	reduce to $n = 4$, 1001, by Lemma 5.7
1	$0 \ldots 0$	$1 \ldots 1$	yes	reduce to $n = 4$, 1001, by Lemma 5.7
$1 \ldots 1$	0	1	no	over the ridge from $\imath$
$1 \ldots 1$	0	$1 \ldots 1$	yes	reduce to $n = 5$, 11011, by Lemma 5.7
$1 \ldots 1$	$0 \ldots 0$	1	yes	reduce to $n = 4$, 1001, by Lemma 5.7
$1 \ldots 1$	$0 \ldots 0$	$1 \ldots 1$	yes	reduce to $n = 5$, 11011, by Lemma 5.7

TABLE 1. List of all sets with two slopes (up to complement).

Three slopes. This case can be tackled using the same methods we applied in the case above. Table 2 gives the results.

$s \geq 4$ *slopes.* Using Lemma 5.7, we reduce such a set to an alternating set with s slopes showing that for all these sets U the pair $\imath \nearrow U$ defines an edge of $\overline{Q_n}$. This is in accordance with the statement of the theorem because sets which are over the ridge from $\imath$ can have at most three slopes. The statement for alternating sets is proven by induction on n in Lemma 5.9 below. Note that the starts of the inductions in the proof of that lemma are $n = 5$ and $n = 6$ for even or odd s respectively.

This concludes the proof of the theorem. □

We now present the inductive construction which we need for the case of an even number of $s \geq 4$ slopes.

Word				Edge?	Why?
Hill 1	Valley 1	Hill 2	Valley 2		
1	0	1	0	no	over the ridge from $\imath$
1	0	1	$0\ldots0$	no	over the ridge from $\imath$
1	0	$1\ldots1$	0	yes	reduce to $n = 5$, 10110, by Lemma 5.7
1	0	$1\ldots1$	$0\ldots0$	yes	reduce to $n = 5$, 10110, by Lemma 5.7
1	$0\ldots0$	1	0	yes	reduce to $n = 5$, 10010, by Lemma 5.7
1	$0\ldots0$	1	$0\ldots0$	yes	reduce to $n = 5$, 10010, by Lemma 5.7
1	$0\ldots0$	$1\ldots1$	0	yes	reduce to $n = 5$, 10010, by Lemma 5.7
1	$0\ldots0$	$1\ldots1$	$0\ldots0$	yes	reduce to $n = 5$, 10110, by Lemma 5.7
$1\ldots1$	0	1	0	no	over the ridge from $\imath$
$1\ldots1$	0	1	$0\ldots0$	no	over the ridge from $\imath$
$1\ldots1$	0	$1\ldots1$	0	yes	reduce to $n = 5$, 10110, by Lemma 5.7
$1\ldots1$	0	$1\ldots1$	$0\ldots0$	yes	reduce to $n = 5$, 10110, by Lemma 5.7
$1\ldots1$	$0\ldots0$	1	0	yes	reduce to $n = 5$, 10010, by Lemma 5.7
$1\ldots1$	$0\ldots0$	1	$0\ldots0$	yes	reduce to $n = 5$, 10010, by Lemma 5.7
$1\ldots1$	$0\ldots0$	$1\ldots1$	0	yes	reduce to $n = 5$, 10010, by Lemma 5.7
$1\ldots1$	$0\ldots0$	$1\ldots1$	$0\ldots0$	yes	reduce to $n = 5$, 10010, by Lemma 5.7

TABLE 2. List of all sets with three slopes (up to complement).

Lemma 5.9. *For an integer $n \geq 5$ let U be an alternating subset of $[n]$. The pair $\imath{\diagup}U$ defines an edge of $\overline{Q_n}$.*

Proof. We first prove the case when n is odd.

The proof is by induction over n. For the start of the induction we consider $n = 5$ and offer the matrix $C^{10101} \in \mathbb{S}_5^0$ in Table 3 of the appendix satisfying (8). We will need this matrix in the inductive construction.

Now set $D^5 := C^{10101}$ and assume that the pair $\imath{\diagup}U^-$ defines an edge of $\overline{Q_n}$ where U^- is an alternating subset of $[n]$. W.l.o.g., we assume that $U^- = 10\ldots01$. There exists a matrix $D^- \in \mathbb{S}_n^0$ for which (8) holds. We will construct a matrix $D \in \mathbb{S}_{n+2}^0$ satisfying (8) for $U := 010\ldots010$.

We extend D^- to a $(n + 2) \times (n + 2)$-Matrix

$$\widehat{D} := \begin{pmatrix} D^- & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^\top & 0 & 0 \\ \mathbf{0}^\top & 0 & 0 \end{pmatrix}.$$

We do the same with D^5, except on the other side:

$$\widehat{D^5} := \begin{pmatrix} 0 & 0 & \mathbf{0}^\top \\ 0 & 0 & \mathbf{0}^\top \\ \mathbf{0} & \mathbf{0} & D^5 \end{pmatrix}.$$

Now we let $D := \widehat{D} + \widehat{D^5}$ and check the conditions (8) on D. These are now easily verified.

For the even case we guarantee the start of induction investigating $n = 6$. We give a matrix C^{101010} satisfying (8) in Table 3 in the appendix. (Note that 101010 is the only set which is not incident to $\imath$, is not over the ridge from $\imath$ or $\imath^-$, cannot be reduced by Lemma 5.7 and is no complement of sets of any of these three types.) The induction is proved in the same way by using the matrix $D^6 := C^{101010}$. $\qquad\square$

Some consequences. From Theorem 5.1, we immediately have the following two corollaries.

Corollary 5.10. *For $n \geq 4$, the number of unbounded edges issuing from a vertex of $\overline{Q_n} = P_n + C_n$ is $2^{n-1} - n$.* $\qquad\square$

Corollary 5.11. *The convex set Q_n is closed if and only if $n \leq 3$.* $\qquad\square$

6. OUTLOOK

Starting from simple observations regarding the set E_n of all semi-metrics which are embeddable in dimension one and their convex hull, we have studied some properties of the set E_n^b of metrics which are embeddable in the real line and which separate any two points by at least a fixed constant $\varepsilon > 0$. While the convex hull of E_n coincides with the cut cone C_n, the closure $\overline{Q_n}$ of the convex hull Q_n of E_n^b is the Minkowski sum $P_n + C_n$. We have given a combinatorial characterization of the unbounded edges of both Q_n and $\overline{Q_n}$.

There are some interesting open question in this context. First of all, it would be interesting to see whether a combinatorial relationship can be found for unbounded faces of higher dimension containing a fixed vertex. Here Proposition 4.6 gives only a partial answer. Further, the question remains whether *bounded* edges have a combinatorial interpretation.

Last but not least, in the context of geometry of semi-metrics, while the set of *all* ℓ_p-embeddable semi-metrics is a convex cone (which is polyhedral for $p = 1$), the set of non-zero semi-metrics embedabble in dimension one is topologically non-trivial in the sense of Proposition 3.4-(b). This suggests that the topology of semi-metrics which are ℓ_p-embeddable in dimension at most d for $1 < d < \binom{n}{2}$ might be an intriguing topic. Studying these sets topologically might even shed some light on old open problems concerning the so-called *minimum ℓ_p-dimension:* This is the smallest d such that any isometrically ℓ_p-embeddable semi-metric can be isometrically embedded in the ℓ_p space of dimension d. While the exact number is easily seen to be $n - 1$ for $p = 2$, it is not known for the other values of p. For example, Ball [2] conjectured that for the minimum ℓ_1-dimension is $\binom{n-2}{2}$.

APPENDIX

n	Slopes		Matrix
4	2	$C^{1001} :=$	$\begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 3 & -2 \\ -2 & 3 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$
5	2	$C^{11011} :=$	$\begin{pmatrix} 0 & 8 & -6 & -1 & -1 \\ 8 & 0 & 2 & 9 & -3 \\ -6 & 2 & 0 & 5 & -7 \\ -1 & 9 & 5 & 0 & 11 \\ -1 & -3 & -7 & 11 & 0 \end{pmatrix}$
5	3	$C^{10110} :=$	$\begin{pmatrix} 0 & 2 & 2 & 1 & -3 \\ 2 & 0 & 0 & -2 & 2 \\ -2 & 0 & 0 & 2 & 0 \\ 1 & -2 & 2 & 0 & 1 \\ -3 & 2 & 0 & 1 & 0 \end{pmatrix}$
5	3	$C^{10010} :=$	$\begin{pmatrix} 0 & 2 & -2 & 2 & -2 \\ 2 & 0 & 4 & -3 & 1 \\ -2 & 4 & 0 & 1 & 1 \\ 2 & -3 & 1 & 0 & 1 \\ -2 & 1 & 1 & 1 & 0 \end{pmatrix}$
5	4	$C^{10101} :=$	$\begin{pmatrix} 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 1 & 1 & -2 \\ 3 & 1 & 0 & 1 & 3 \\ -2 & 1 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 0 \end{pmatrix}$
6	5	$C^{101010} :=$	$\begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 1 & 3 & -2 \\ -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{pmatrix}$

TABLE 3. Matrices certifying unbounded edges of Q_n

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